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3 (Sem-4/CBCS) MAT HC 3

2024

MATHEMATICS

(Honours Core)

Paper : MAT-HC-4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : 1×10=10
- (a) Give an example to show that for two non-zero elements a and b of a ring R , the equation $ax = b$ can have more than one solution.
- (b) How many nilpotent elements have in an integral domain ?

Contd.

(c) Which of the following statements is not true ?

(i) $\langle 5 \rangle$ is a prime ideal of Z .

(ii) $\langle 5 \rangle$ is a maximal ideal of Z .

(iii) $\langle 5 \rangle$ is a maximal ideal of Z_{20} .

(iv) $\frac{Z}{5Z}$ is an integral domain.

(d) Define prime ideal of a ring.

(e) Give example of a commutative ring without zero divisor that is not an integral domain.

(f) Consider the polynomial

$$f(x) = 4x^3 + 2x^2 + x + 4 \text{ and}$$

$$g(x) = 3x^4 + 3x^3 + 3x^2 + x + 4 \text{ in } Z_5.$$

Compute $f(x) + g(x)$.

(g) Write $f(x) = x^3 + x^2 + x + 1 \in Z_2[x]$ as a product of irreducible polynomial over Z_2 .

(h) Which of the following is a primitive polynomial ?

(i) $2x^3 + 4x^2 + 6x + 10$

(ii) $5x^2 - 30x - 20$

(iii) $2x^4 + 3x^3 + 5x^2 - 7x + 11$

(iv) $3x^2 - 3x + 3$

(i) State whether the following statement is true or false :

“A polynomial $f(x)$ in $Z[x]$ which is reducible over Z is also reducible over Q .”

(j) Choose the correct statement :

(i) Every Euclidean domain is a unique factorization domain.

(ii) Every principal ideal domain is a Euclidean domain.

(iii) Every unique factorization domain is a Euclidean domain.

(iv) Every unique factorization domain is a principal ideal domain.

2. Answer the following questions : $2 \times 5 = 10$

- (a) If a and b are two idempotents in a commutative ring, then show that $a+b-ab$ is also an idempotent element.
- (b) Show that every non-zero element of Z_n is a unit or a zero divisor.
- (c) Show that every ring homomorphism $f: Z_n \rightarrow Z_n$ is of the form $f(x) = ax$ where $a = a^2$.
- (d) Find the zeros of $f(x) = x^2 + 3x + 2$ in Z_6 .
- (e) Let D be an integral domain and $a, b \in D$. If $\langle a \rangle = \langle b \rangle$, then show that a and b are associates.

3. Answer **any four** questions : $5 \times 4 = 20$

- (a) The operations \oplus and \otimes defined on the set Z of integers by $a \oplus b = a + b - 1$ and $a \otimes b = a + b - ab$. Show that (Z, \oplus, \otimes) is a ring with unity.
- (b) Find all ring homomorphism from $Z \oplus Z$ to Z .

(c) Let R be a commutative ring with unity. Show that an ideal A of R is prime if and only if the quotient ring $\frac{R}{A}$ is an integral domain.

(d) Define principal ideal domain. Show that if F is a field, then $F[x]$ is a principal ideal domain. $1+4=5$

(e) Show that every Euclidean domain is a principal ideal domain.

(f) Show that the number of reducible polynomials over Z_p of the form $x^2 + ax + b$ is $\frac{p(p+1)}{2}$.

4. Answer the following questions : $10 \times 4 = 40$

(a) (i) Let R be a commutative ring with unity. Show that the set

$$R[x] = \{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 / a_i \in R,$$

n is a non-negative integer} is a ring. Also show that if R is an integral domain, then $R[x]$ is also an integral domain. $5+2=7$

- (ii) Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $Z_7[x]$.

Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$. 3

Or

- (i) Show that

$$Z[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in Z\} \text{ and}$$

$$H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} / a, b \in Z \right\}$$

are isomorphic as ring. 4

- (ii) If a, b be any two ring elements and m and n be any two integers, then show that $(m.a)(n.b) = (mn).(ab)$ 6

- (b) (i) Define maximal ideal of a ring. Let A be an ideal of a commutative ring with unity R . Prove that $\frac{R}{A}$ is a field if and only if A is maximal. 1+6=7

- (ii) Let R be a commutative ring and A be any subset of R . Show that the nil-radical of A ,

$$N(A) = \{r \in R / r^n \in A \text{ for some } n \in N\}$$

is an ideal of R . 3

Or

- (i) Let ϕ be a ring homomorphism from R to S . Then the mapping

from $\frac{R}{\ker(\phi)}$ to $\phi(R)$, given by

$r + \ker(\phi) \rightarrow \phi(r)$ is an

isomorphism, i.e., $\frac{R}{\ker(\phi)} \approx \phi(R)$. 6

- (ii) Let ϕ be a ring homomorphism from a ring R to a ring S . Let B be an ideal of S . Then $\phi^{-1}[B] = \{r \in R : \phi(r) \in B\}$ is an ideal of R . 4

- (c) If F is a field and $p(x) \in F[x]$, then prove that $\frac{F[x]}{\langle p(x) \rangle}$ is a field if and only if $p(x)$ is irreducible over F .

Or

Let F be a field. If $f(x) \in F[x]$ and degree $f(x)$ is 2 or 3, then $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F . Is the result true when degree $f(x)$ is greater than 3? Justify.

7+3=10

- (d) In a principal ideal domain, show that an element is irreducible if and only if it is prime. Use this result to show that $Z[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in Z\}$ is not a principal ideal domain.

7+3=10

Or

- (i) In a principal ideal domain, show that any strictly increasing chain of ideals $I_1 \subset I_2 \subset \dots$ must be finite in length.

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- (ii) Let ϕ be a onto ring homomorphism from a ring R to a ring S . Then prove that ϕ is an isomorphism if and only if $\ker(\phi) = \{0\}$.

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- (iii) Determine all ring homomorphism from the ring of integers Z to itself.

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