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3 (Sem–3/CBCS) MAT HC 2

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–3026

(Group Theory–I)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1 \times 10 = 10$

   (a) "The set S of positive irrational numbers together with 1 is a group under multiplication." Justify whether it is true or false.

   (b) "In the set of integers, subtraction is not associative." Justify the statement.

   (c) "Product of two subgroups of a group is again a subgroup." State whether true or false.

(d) In the group $\mathbb{Z}_{12}$, find the order of 6.

(e) Write all the generators of $\mathbb{Z}_8$.

(f) List all the elements of the group $\frac{\mathbb{Z}}{4\mathbb{Z}}$.

(g) Give the statement of Cayley's theorem.

(h) Write the following permutation as product of 2-cycles :

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 4 & 6 & 5 & 7 & 2 \end{pmatrix}$$

(i) State whether the following statement is true or false :

"If the homomorphic image of a group is Abelian, then the group itself is Abelian."

(j) Give the statement of second isomorphism theorem.

2. Answer the following questions : 2×5=10

(a) Prove that in a group G, for any elements a and b and any integer n, $(a^{-1}ba)^n = a^{-1}b^n a$.

(b) Show that in a group G, right and left cancellation laws hold.

(c) Define centre of a group G and give an example.

(d) What is meant by cycle of a permutation ? Give an example.

(e) If ϕ is a homomorphism from a group G onto a group $\overline{G}$, then show that ϕ carries the identity element of G to the identity element of $\overline{G}$.

3. Answer ***any four*** questions : 5×4=20

(a) Let G be a group and H be a non-empty finite subset of G. Prove that H is a subgroup of G if and only if H is closed under the operation in G.

(b) Let G be a group and H be a subgroup of G. For an element a in G, prove that $aH = H$ if and only if a is in H.

(c) Let G be a finite group and H be a subgroup of G. Prove that $|H|$ divides $|G|$.

(d) Prove that in a finite group, the number of elements of order d is divisible by $\phi(d)$.

(e) Define external direct product of a finite collection of groups. List all elements of $U(8) \oplus U(10)$ and find $|U(8) \oplus U(10)|$. 2+2+1=5

(f) Let ϕ be a homomorphism from a group G to a group $\bar{G}$. If $\bar{K}$ is a normal subgroup of $\bar{G}$, prove that $\phi^{-1}(\bar{K}) = \{k \in G \mid \phi(k) \in \bar{K}\}$ is a normal subgroup of G.

*Answer **either** (a) **or** (b) from each of the following questions (Q.4 to Q.7) :* 10×4=40

4. (a) Describe the elements of D_4, the symmetrics of a square. Write down a complete Cayley's table for D_4. Show that D_4 forms a group under composition of functions. Is D_4 an Abelian group? 2+3+4+1=10

(b) Let G be a group and H be a non-empty subset of H. Prove that H is a subgroup of G if and only if $a.b^{-1}$ is in H whenever a and b are in H. Also, write all the subgroups of the group of integers $\mathbb{Z}$. 8+2=10

5. (a) (i) Let a be an element of order n in a group G and let k be a positive integer. Then prove that

$$\langle a^k \rangle = \langle a^{gcd(n,k)} \rangle \text{ and}$$

$$|a^k| = \frac{n}{gcd(n,k)}$$

(ii) Prove that in a finite cyclic group, the order of an element divides the order of the group. 8+2=10

(b) Prove that every subgroup of a cyclic group is cyclic. Also, if $|\langle a \rangle| = n$, then show that the order of any subgroup of $\langle a \rangle$ is a divisor of n; and for each positive divisor k of n, the group $\langle a \rangle$ has exactly one subgroup of order k, which is $\langle a^{n/k} \rangle$. 5+2+3=10

6. (a) Prove that every group is isomorphic to a group of permutations.

(b) Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.

7. (a) (i) Let ϕ be a group homomorphism from a group G to a group $\bar{G}$. Prove that $\frac{G}{Ker\,\phi}$ is isomorphic to $\phi(G)$.

(ii) Prove that $\frac{\mathbb{Z}}{\langle n\rangle} \cong \mathbb{Z}_n$. 7+3=10

(b) Let ϕ be an isomorphism from a group G onto a group $\bar{G}$. Prove that :

(i) for every integer n and for every group element a in G, $\phi(a^n) = [\phi(a)]^n$.

(ii) $|a| = |\phi(a)|$ for all a in G.

(iii) if G is finite, then G and $\bar{G}$ have exactly the same number of elements of every order.

4+3+3=10