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3 (Sem-4/CBCS) MAT HC 3

2025

MATHEMATICS

(Honours Core)

Paper : MAT-HC-4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : 1×10=10

- (a) Define characteristic of a ring.
- (b) State whether the following statement is *True or False* :
“ $2\mathbb{Z} \cup 3\mathbb{Z}$ is a subring of \mathbb{Z} ”.
- (c) In the ring of integers, find a positive integer x such that $\langle x \rangle = \langle m \rangle + \langle n \rangle$.

- (d) How many zeros does the equation $x^2 + 3x + 2 = 0$ have in \mathbb{Z}_6 ?
- (e) Find all the maximal ideals in \mathbb{Z}_8 .
- (f) What is the characteristic of $\mathbb{Z}_m \oplus \mathbb{Z}_n$?
- (g) Is the polynomial $x^2 + x + 4$ irreducible over \mathbb{Z}_{11} ?
- (h) State whether the following statement is **True or False** :
 " \mathbb{Z}_6 is a subring of \mathbb{Z}_{12} ".
- (i) Define prime ideal of a ring.
- (j) Given an example of a UFD, which is not a PID.

2. Answer the following questions : $2 \times 5 = 10$

- (a) Show that the centre of a ring is a subring.
- (b) Prove that the only ideals of a field \mathbb{F} are $\{0\}$ and \mathbb{F} itself.
- (c) Is the mapping $\phi : \mathbb{Z}_5 \rightarrow \mathbb{Z}_{30}$ given by $\phi(x) = 6x$ a ring homomorphism ?

- (d) Consider $f(x) = x^3 + 2x + 4$ and $g(x) = 3x + 2$ in $\mathbb{Z}_5[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

- (e) Let $f(x) = x^3 + x^2 + x + 1 \in \mathbb{Z}_2[x]$. Write $f(x)$ as a product of irreducible polynomials over \mathbb{Z}_2 .

3. Answer **any four** questions : $5 \times 4 = 20$

- (a) Let x be a positive integer. Show that $Q[\sqrt{x}] = \{a + b\sqrt{x} : a, b \in Q\}$ is a field.
- (b) Let R be a commutative ring with unity. Show that an ideal A of R is prime if and only if the quotient ring R/A is an integral domain.
- (c) Define integral domain. Prove that if D is an integral domain, then the polynomial ring $D[x]$ is also an integral domain. $1 + 4 = 5$
- (d) Show that every Euclidean domain is a principal ideal domain.

(e) Show that $x^4 + 1$ is irreducible over \mathbb{Q} but reducible over \mathbb{R} .

(f) Let D be a PID and let $p \in D$. Prove that $\langle p \rangle$ is maximal in D if and only if p is irreducible.

4. Answer the following questions : $10 \times 4 = 40$

(a) (i) Define subring. Prove that a nonempty subset S of a ring R is a subring. If S is closed under subtraction and multiplication.

$1+5=6$

(ii) Let $R = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $S = \{(a, b, c) \in R : a + b = c\}$. Prove or disprove that S is a subring of R .

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Or

(i) Let R be a finite commutative ring with unity. Prove that every nonzero element of R is either a zero-divisor or a unit.

5

(ii) Describe all zero-divisors and units of $\mathbb{Z} \oplus \mathbb{Q} \oplus \mathbb{Z}$.

5

(b) (i) Let ϕ be a ring homomorphism from R to S . Then the mapping from $R/\text{Ker } \phi$ to $\phi(R)$, given by $r + \text{Ker } \phi \rightarrow \phi(r)$ is an isomorphism, i.e., $R/\text{Ker } \phi \cong \phi(R)$.

6

(ii) Let $S = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$. Show

that $\phi: \mathbb{C} \rightarrow S$ given by

$\phi(a + ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a ring

isomorphism.

4

OR

For a field \mathbb{F} , define and prove the division algorithm for $\mathbb{F}[x]$.

$2+8=10$

(c) (i) Prove that $\mathbb{Z}[\sqrt{5}]$ is not a unique factorization domain.

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- (ii) Show that the ring of Gaussian integers $\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}\}$ is a Euclidean domain with

$$d(a + ib) = a^2 + b^2. \quad 6$$

OR

- (i) Find all units, zero-divisors, idempotents and nilpotent elements in $\mathbb{Z}_3 \times \mathbb{Z}_6$. 6

- (ii) Let \mathbb{F} be a field of prime characteristic p . Prove that $K = \{x \in \mathbb{F} : x^p = x\}$ is a sub field of \mathbb{F} . 4

- (d) (i) In $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients, let $I = \{f(x) \in \mathbb{Z}[x] : f(0) = 0\}$. Prove that $I = \langle x \rangle$. 5

- (ii) Show that $\mathbb{R} / \langle x^2 + 1 \rangle$ is a field. 3

- (iii) Show that the kernel of a homomorphism is an ideal. 2

OR

- (i) Let \mathbb{F} be a field, then show that $\mathbb{F}[x]$ is a principal ideal domain. 5

- (ii) Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \in \mathbb{Z}(x)$$

If there is a prime p such that $p \nmid a_n$, $p \mid a_{n-1}, \dots, p \mid a_0$ and $p^2 \nmid a_0$. Then show that $f(x)$ is irreducible over \mathbb{Q} . 5