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3 (Sem-6/CBCS) MAT HC 1 (N/O)

2025

MATHEMATICS

(Honours Core)

Paper : MAT-HC-6016

[New Syllabus]

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

[Old Syllabus]

(Complex Analysis)

Full Marks : 60

Time : Three hours

***The figures in the margin indicate
full marks for the questions.***

[New Syllabus]

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

1. Answer the following as directed :

1×10=10

(a) A bounded function $f: [a, b] \rightarrow \mathbb{R}$ is integrable if for each $\varepsilon > 0$, there exists a partition P such that

(i) $U(f, P) < \varepsilon + L(f, P)$

(ii) $U(f, P) < \varepsilon - L(f, P)$

(iii) $U(f, P) > \varepsilon + L(f, P)$

(iv) $U(f, P) > \varepsilon - L(f, P)$

(Choose the correct option)

(b) State mean value theorem for integrals.

(c) Evaluate $\Gamma \frac{3}{2}$.

(d) Define Euclidean metric on \mathbb{R}^n .

(e) The open ball $S\left(\frac{1}{2}, 1\right)$ on the usual metric space (\mathbb{R}, d) is

(i) $\left(\frac{1}{2}, \frac{3}{2}\right)$

(ii) $\left(\frac{1}{2}, -\frac{3}{2}\right)$

(iii) $\left(-\frac{1}{2}, \frac{3}{2}\right)$

(iv) $\left(-\frac{1}{2}, -\frac{3}{2}\right)$

(Choose the correct option)

(f) Let X be a non-empty set. If $d: X \times X \rightarrow \mathbb{R}$ is a pseudometric on X , then which of the following statement is false ?

(i) $d(x, y) \geq 0$ for all $x, y \in X$

(ii) $d(x, y) = 0 \Rightarrow x = y$ for all $x, y \in X$

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all

$x, y, z \in X$

(Choose the correct option)

(g) If A is a non-empty subset of a metric space (X, d) such that A^c is closed in X , then A is

- (i) closed in X
- (ii) open in X
- (iii) Both open and closed in X
- (iv) None of the above

(Choose the correct option)

(h) Show that the closure \bar{F} of $F \subseteq X$, where (X, d) is a metric space, is closed.

(i) Define a contraction mapping on a metric space.

(j) Which of the following statements are true?

- (i) A singleton set $\{x\}$ in any metric space is always connected.
- (ii) The interval $[2, 3]$ is not connected in the usual metric space (\mathbb{R}, d) .
- (iii) If (X, d) is a connected metric space, there exists a proper subset of X which is both open and closed in X .

(iv) Closure of a connected set in a metric space is connected.

(Choose the correct option)

2. Answer the following questions : $2 \times 5 = 10$

(a) Let $f(x) = x$ on $[0, 1]$ and

$$P = \left\{ x_i = \frac{i}{8}, i = 0, 1, 2, \dots, 8 \right\}$$

Find $L(f, P)$ and $U(f, P)$

(b) Prove that $\overline{(\alpha + 1)} = \alpha \overline{\alpha}$

(c) Show that the discrete metric space is a complete metric space.

(d) Let (X, d) be a metric space and

$\bar{S}(x, r) = \{y \in X : d(x, y) \leq r\}$ be a closed ball in X . Prove that $\bar{S}(x, r)$ is closed.

(e) Prove that if Y is a connected set in a metric space (X, d) , then any set Z such that $Y \subseteq Z \subseteq \bar{Y}$ is connected.

3. Answer **any four** questions : $5 \times 4 = 20$

(a) Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Prove that f is integrable.

(b) Show that $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{2n+k} = \log \frac{3}{2}$

(c) Define an open ball in a metric space. Prove that in any metric space (X, d) , each open ball is an open set. $1+4=5$

(d) Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that a mapping $f : X \rightarrow Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y .

(e) A continuous function may not map a Cauchy sequence into a Cauchy sequence — Justify it.

Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \rightarrow Y$ be uniformly continuous. If $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in X , then show that $\{f(x_n)\}_{n \geq 1}$ is also a Cauchy sequence in Y . $1+4=5$

(f) Let (X, d_X) be a metric space. If every continuous function

$f : (X, d_X) \rightarrow (\mathbb{R}, d)$ has the intermediate value property, then prove that (X, d_X) is a connected metric space.

Answer **either** (a) **or** (b) of the following questions : (Q.4 to Q.7) $10 \times 4 = 40$

4. (a) (i) State and prove First Fundamental Theorem of Calculus. $1+4=5$

(ii) Discuss the convergence of the integral $\int_1^\infty \frac{1}{x^p} dx$ for various values of p . 5

(b) (i) Show that $f : [0,1] \rightarrow \mathbb{R}$ defined by $f(x) = x^n$ is integrable and $\int_0^1 f(x) dx = \frac{1}{n+1}$. 4

(ii) Let f be continuous on $[a, b]$. Prove that there exists $c \in [a, b]$ such that $\frac{1}{b-a} \int_a^b f(x) dx = f(c)$.

Use the 1st mean value theorem to prove that for $0 < a \leq 1$ and

$$n \in \mathbb{N}, \int_0^1 \frac{x^n}{1+x} dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$3+3=6$

5. (a) (i) Let $X = \mathbb{R}$. For $x, y \in \mathbb{R}$, define

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}. \text{ Show that } d$$

is a metric on \mathbb{R} . 4

- (ii) Prove that a convergent sequence in a metric space is a Cauchy sequence.

Does the converse of this hold? Justify it. 4+2=6

- (b) (i) Prove that the metric space $X = \mathbb{R}^n$ with the metric given by

$$d_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1$$

where $x = (x_1, x_2, \dots, x_n)$ and

$y = (y_1, y_2, \dots, y_n)$ are in \mathbb{R}^n , is a complete metric space. 5

- (ii) Let (X, d) be a metric space and F_1, F_2 be subsets of X . Prove

that $(F_1 \cup F_2)' = F_1' \cup F_2'$ and

$$\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}. \quad 3+2=5$$

6. (a) (i) Let (X, d) be a metric space and let $x \in X$ and $A \subseteq X$ be non-empty. Then prove that $x \in \bar{A}$ if and only if $d(x, A) = 0$. 4

- (ii) Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subseteq X$. Prove that a function $f : A \rightarrow Y$ is continuous at $a \in A$ if and only if whenever a sequence $\{x_n\}$ in A converges to a , the sequence $\{f(x_n)\}$ converges to $f(a)$. 6

- (b) (i) Prove that a mapping $f : X \rightarrow Y$ is continuous on X iff $f^{-1}(F)$ is closed in X for all closed subsets F of Y . 4

- (ii) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \rightarrow Y$. Prove that the following statements are equivalent :

I. f is continuous on X

II. $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all
 $B \subseteq Y$

III. $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$

6

7. (a) Let (\mathbb{R}, d) be the space of real numbers with the usual metric. Prove that a subset $I \subseteq \mathbb{R}$ is connected if and only if I is an interval. 10

(b) (i) If f and g are two uniformly continuous mappings of metric spaces (X, d_X) to (Y, d_Y) and (Y, d_Y) to (Z, d_Z) respectively, then prove that $g \circ f$ is uniformly continuous mapping of (X, d_X) to (Z, d_Z) .

Show that the function
 $f : (0, 1) \rightarrow \mathbb{R}$ defined by

$f(x) = \frac{1}{x}$ is not uniformly continuous. 4+2=6

(ii) Let (X, d) be a metric space and let $\{Y_\lambda : \lambda \in \Lambda\}$ be a family of connected sets in (X, d) having a nonempty intersection. Prove that

$Y = \bigcup_{\lambda \in \Lambda} Y_\lambda$ is connected. 4

[Old Syllabus]

(Complex Analysis)

Full Marks : 60

Time : Three hours

1. Answer the following questions : $1 \times 7 = 7$

(a) Write down the Cauchy-Riemann equations.

(b) Define analytic function.

(c) Find the argument of $\frac{1-i}{1+i}$.

(d) If $z_1 = 2+i$ and $z_2 = 3-2i$, then evaluate $|3z_1 - 4z_2|$.

(e) Find $\lim_{z \rightarrow i} (z^2 + 2z)$.

(f) Find $((3-i)^2 - 3)i$.

(g) Express $e^{-i\frac{\pi}{4}}$ in the form $a+bi$.

2. Answer the following questions : $2 \times 4 = 8$

(i) Write $\frac{1-i}{3}$ in the form $re^{i\theta}$.

(ii) Find $\left| \frac{1+2i}{-2-i} \right|$.

(iii) Determine the points at which the function $\frac{1}{z-2+3i}$ is not analytic.

(iv) For any two complex numbers z_1 and z_2 , prove that $|z_1 z_2| = |z_1| |z_2|$.

3. Answer **any three** questions : $5 \times 3 = 15$

(a) Prove that $f(z) = z^2 - 2z + 5$ is continuous everywhere in the finite plane.

(b) Show that $f(z) = e^z$ is analytic at every point of the complex plane.

(c) Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^z}{z-2}$, where C is the circle $|z|=1$.

(d) If $f(z) = z^3 - 2z$; $z \in \mathbb{C}$, then find $f'(z)$ at $z = -1$, provided the value exists.

- (e) Let $f(z) = \begin{cases} z^2, & z \neq i \\ 0, & z = i \end{cases}$, prove that $f(z)$ is not continuous at $z = i$.

4. Answer **any three** questions : $10 \times 3 = 30$

- (i) Prove that the necessary and sufficient conditions for the complex function $\omega = f(z) = u(x, y) + iv(x, y)$ to be analytic in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where all partial derivatives are assumed to be continuous on R .

- (ii) If $f(z)$ is analytic with its derivative $f'(z)$ continuous at all points inside and on a simple closed curve C , prove that $\int_C f(z) dz = 0$.

- (iii) Prove that if $f(z)$ is integrable along a curve C having finite length L and if there exists a positive number M such that $|f(z)| \leq M$ on C , then

$$\left| \int_C f(z) dz \right| \leq ML.$$

- (iv) (a) Find the analytic function whose real part is

$$u = e^{-x} \left[(x^2 - y^2) \cos y + 2xy \sin y \right].$$

5

- (b) Show that the function

$$f(z) = \sin x \cosh y + i \cos x \sinh y$$

is entire.

5

- (v) (a) State and prove Cauchy's Integral Formulae.

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- (b) Evaluate $\frac{1}{2\pi i} \int_C \frac{e^z}{z-2} dz$, where C is

the circle $|z| = 3$.

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(vi) (a) Suppose that

$$z_n = x_n + iy_n, (n = 1, 2, 3 \dots) \text{ and}$$

$$z = x + iy. \text{ Prove that } \lim_{n \rightarrow \infty} z_n = z$$

if and only if $\lim_{n \rightarrow \infty} x_n = x$ and

$$\lim_{n \rightarrow \infty} y_n = y. \quad 5$$

(b) Show that,
$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n,$$

$$(|z| < \infty). \quad 5$$